Combating Online Counterfeits: A Game-Theoretic Analysis of Cyber Supply Chain Ecosystem ⋆

Yuhan Zhao[1] and Quanyan Zhu[1]

New York University, Brooklyn NY 11201, USA
{yhzhao,qz494@nyu.edu}@nyu.edu

Abstract. Counterfeiting has been a pervasive threat to the security of supply chains. With the development of cyber technologies, traditional supply chains move their logistics to the cyberspace for better efficiency. However, counterfeiting threats still exist and may even cause worse consequences. It is imperative to find mitigating strategies to combat counterfeiting in the cyber supply chain. In this paper, we establish a games-in-games framework to capture the interactions of counterfeiting activities in the cyber supply chain. Specifically, the sellers in the cyber supply chain play a Stackelberg game with consumers, while sellers compete with each other by playing a Nash game. All sellers and consumers aim to maximize their utilities. We design algorithms to find the best response of all participants and analyze the equilibrium of the supply chain system. Finally, we use case studies to demonstrate the equilibrium behavior and propose effective anti-counterfeit strategies.

Keywords: Game Theory · Cyber Supply Chain · Supply Chain Security · Games-in-Games Framework · Anti-Counterfeit Strategy

1 Introduction

A supply chain is a network that integrates business entities, information, and resources to produce and distribute a specific product to final consumers [2, 13, 15]. One of the significant threats to supply chain security is counterfeiting. Counterfeits can be roughly categorized into *deceptive* and *non-deceptive* counterfeits. Deceptive counterfeits can penetrate the licit supply chain in intermediate processes such as manufacturing and distribution, directly disrupting the market order and damaging the market regulations. Non-deceptive counterfeits, on the other hand, influence the market through the illicit supply chain, which opens a way for counterfeit trafficking. The non-deceptive counterfeits circulate to consumers via the illicit market and produce an indirect negative impact on the

⋆ This research is partially supported by award 2015-ST-061-CIRC01, U. S. Department of Homeland Security, awards ECCS-1847056, CNS-1544782, CNS-2027884, and SES-1541164 from National Science of Foundation (NSF), and grant W911NF-19-1-0041 from Army Research Office (ARO).

licit market by occupying the market share that belongs to the genuine goods. To summarize, counterfeiting can pose security threats to the supply chain both *explicitly* and *implicitly* by deceptive and non-deceptive counterfeits.

With the development of cyber technologies, many corporations have moved the operation of their supply chains to the cyberspace by developing online markets. Despite the merits of convenience and efficiency, such a paradigm shift does not address counterfeiting issues in the cyber supply chain. It can even exacerbate counterfeiting threats due to the new features of the cyber supply chain.

One notable feature of the cyber supply chains is its simplicity. While the traditional supply chain contains sophisticated intermediate processes such as inventory and transportation to ensure the completeness, the structure of the cyber supply chain is more straightforward. The postal services replace the distribution process, and the retail takes place online instead of in physical stores. Therefore, sellers and buyers can make direct contact through online markets. The lack of intermediate processes in the cyber supply chain makes it even harder for inspections. For example, the massive amount of packages in postal services challenge the authority to trace counterfeits. The inspection deficiency may result in more severe counterfeiting in supply chains.

Another feature of the cyber supply chain is its accessibility. Cyber technologies enable a large population to conveniently access the cyber supply chain, resulting in a large potential market scale. However, easy accessibility may worsen the counterfeiting issues. First, the identity of participants in the cyber supply chain can be anonymous. The anonymity and indifference facilitate the vending of counterfeits. Second, the illicit market scale becomes non-negligible. The traditional illicit market highly depends on geographical locations. Nevertheless, the advent of online markets and cyber supply chains break geographical location limitations. People can buy counterfeits online, which results in a significant expansion of the illicit market scale.

Due to these new features, counterfeiting in the cyber supply chain has created a series of losses in various industries [7, 14]. Effective policies are imperative to reduce counterfeiting losses. However, some seemingly reasonable and direct strategies may not be sufficient and, at times, counter-productive. An example is the opioid supply chain [17]. The misuse of opioids has caused a growing number of overdose deaths, forcing medical and public health to reduce prescription opioids from 2010 to 2016. However, the reduced supply increased demand for illicit opioids and resulted in the prosperity of the opioid illicit market, especially the illicit online market, causing even more overdose deaths. The failure of such a seemingly reasonable policy is because of the lack of understanding of the ecosystem of the coexisting illicit and licit supply chains.

The concept of equilibrium from game theory can be used to understand the interaction and balanced behaviors of multiple players. Therefore, we propose a games-in-games framework to capture the features of the coexisting cyber supply chains and model the interaction of counterfeiting activities. The proposed framework enables a holistic understanding of how counterfeiting in the cyber supply chain works and the development of effective anti-counterfeit policies. In our model, we aggregate the large population of sellers and consumers into single players to focus on the population level interaction. To characterize the consumers' diversity, we adopt a random variable to represent the consumers' attitude towards the counterfeits. The sellers are abstracted into three individual players: one licit seller and two illicit sellers, who only sell deceptive and non-deceptive counterfeits, respectively. All of them seek to occupy the market share and maximize their utilities. The consumers decide which market to buy the product to maximize their utility. The interactions between the sellers and the consumers can be modeled by a Stackelberg game, and all the sellers' interactions can be represented by a Nash game. We integrate the Nash game and the Stackelberg game into a games-in-games framework. The proposed framework provides an approach for understanding the interactions of counterfeiting activities in the cyber supply chain. It also enables us to simultaneously capture the explicit and implicit impacts of counterfeiting on the cyber supply chain. Through the analysis, we discover the key factors that affect the equilibrium of the counterfeit ecosystem and propose effect population-level anti-counterfeit strategies to overcome counterfeiting in the cyber supply chain.

The contributions of this paper are summarized as follows:

(i) We identify the unique features of the cyber supply chain and propose a games-in-games framework to characterize counterfeiting activities in the cyber supply chains.
(ii) We formulate the counterfeiting problem as a game with piece-wise continuous utilities and develop computational algorithms for this class of games.
(iii) We analyze the impact of different factors on the equilibrium of the counterfeiting problem and propose effective anti-counterfeit strategies to suppress counterfeiting in the cyber supply chain.

1.1 Related Work

Counterfeiting has posed security threats to the supply chain infrastructure. Many recent works have focused on the modeling of the counterfeit supply chains to provide a fundamental understanding of the impact of counterfeiting. Li and Yi [12] have reviewed counterfeiting in supply chains, identifying the impact of counterfeiting, possible producers' reactions, and supply chain structure under counterfeiting issues. Works such as [11, 1, 8] have provided a general analysis of the counterfeiting in supply chains and their impact on the producers and consumers. Studies also examined possible anti-counterfeit methods to mitigate counterfeiting activities. Anti-counterfeit detection and mitigation strategies have been studied in [10, 3]. In particular, Grossman et al. in [9] have categorized counterfeit products into deceptive and non-deceptive ones, and analyzed the impact of foreign non-deceptive counterfeits to the domestic welfare. They have also discussed the enforcement and confiscation policies to fight counterfeits. In the more recent work [18], Zhang et al. have analyzed the strategies to fight counterfeiting when there are one brand name product and a non-deceptive counterfeit in the market. The authors have also studied the fighting strategies equilibrium in the market with two competing brand name product and one counterfeit.

Game-theoretic approaches have been extensively used in supply chain studies to develop strategic solutions to combat counterfeits. The strategic pricing mechanism in the supply chain with counterfeiter and defective items have been studied in [4, 16]. In particular, Cho et al. in [5] have studied the strategic deceptive and non-deceptive counterfeiters' behaviors in licit and illicit supply chain separately. The authors have also analyzed the impact of counterfeiting on the brand-name company and consumers' welfare, and have provided viable strategies to combat counterfeiting. In this work, we adopt a similar consumer model that views consumers as a continuum. The proposed game-theoretic framework focuses on the analysis of the ecosystem consisting of a licit supplier, an illicit online supplier, and consumers, and studies the outcomes of the games between licit and illicit suppliers who anticipate the consumer demands. We observe a phenomenon of oversupply in the market when the licit supplier competes with the illicit supplier in the markets. The analysis of the ecosystem leads to a design of mitigation strategies to mitigate the impact of the illicit products.

1.2 Organization of the Paper

The rest of the paper is organized as follows. Section 2 models all the participants in the licit and illicit supply chains. Section 3 formulates the counterfeiting problem using the games-in-games framework. We analyze the problem and present algorithms to find the best response of each player and the equilibrium solution in Section 4. Case studies are used in Section 5 to demonstrate the algorithms and the equilibrium solution. Several practical anti-counterfeit strategies based on the interaction model are also proposed. Section 6 concludes the paper.

2 Model of Cyber Supply Chain

We focus on the counterfeits of substitute products in the cyber supply chain. These products are more prone to counterfeiting, and the cyber supply chain related to these products suffers both explicit and implicit influence from counterfeiting simultaneously. Regarding the straightforward structure and accessibility features, we aggregate the massive participants in the cyber supply chain into four players, and abstract the interaction of counterfeiting activities in Fig. 1.

The licit seller S_1 manufactures and vends genuine goods with quantity q_1 to the licit market[1]. The illicit seller S_2 (S_3) fabricates deceptive (non-deceptive) counterfeits with quantity q_2 (q_3) and sells them to the licit (illicit) market. The notation in the paper is summarized in Table 1.

[1] We will use licit and illicit markets to refer to the licit and illicit online markets for simplicity. The same applies to the licit and illicit cyber supply chains.

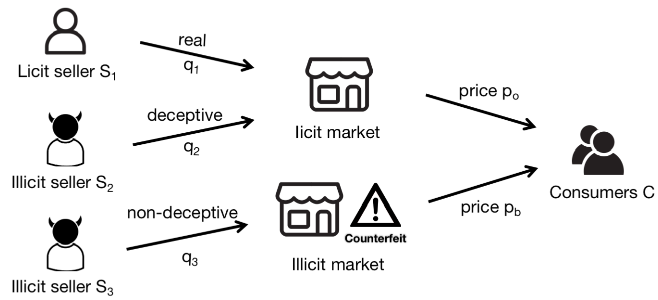


Fig. 1: Sketch of the interaction of counterfeiting activities. We use S_1, S_2, and S_3 to denote the licit seller and two illicit sellers, and C to denote consumers.

Table 1: Summary of notations.

q_i:	the production size of the seller S_i (units);
u_i:	the utility of the seller S_i (\$);
p_o, p_b:	the licit and illicit market sale price per unit (\$/unit);
D_o, D_b:	the licit and illicit market share (or demand) (units);
ϕ_g, ϕ_f:	the consumers' valuation for the real and fake product per unit (\$/unit);
u_c:	the consumers' utility (\$);
a_o, a_b:	the sale price elasticity in the licit and illicit market (\$/unit);
b_o, b_b:	the maximum price in the licit and illicit market (\$);
c_i:	the marginal cost for the seller S_i (\$/unit);
θ:	the type of each consumer;
k:	the scaling factor in consumers' valuation (\$/unit);
a:	the consumers' valuation elasticity;
α:	the weight of price-taking strategy in the illicit market;
η:	the portion of the non-vigilant consumers in the licit market;
γ:	the confiscation probability of the illicit seller in the licit market;
s:	the confiscation penalty of the illicit seller (\$).

2.1 Consumers' Model

Each consumer is characterized by her type θ, a random variable representing her interest in the real and fake product. Consumers with small θ care less about product authenticity because counterfeits may be acceptable substitutes for them. The consumer with $\theta = 0$ has the same interest in the real and fake products. As θ increases, the consumer cares more about product authenticity and gives less valuation to counterfeits. We assume that θ has a uniform distribution over $[0, 1]$, and that the total market scale is 1. A consumer's valuations on a genuine product ϕ_g and a counterfeit ϕ_f are

$$\phi_g = k \cdot 1, \quad \phi_f = k(1 - a\theta), \tag{1}$$

where $k > 0$ is the scaling factor, and $a > 0$ is the valuation elasticity. A consumer has three available actions to maximize her utility: going to the licit market (L), or going to the illicit market (I), or buying nothing (N), denoted as $\mathcal{A}_c = \{L, I, N\}$. When a consumer decides to buy nothing, her utility is trivially zero, *i.e.*, $u_{\theta n} = 0$. The utility for going to the illicit market is

$$u_{\theta b} = \phi_f - p_b. \tag{2}$$

We assume that some of the consumers who choose the licit market are vigilant and skeptical. They prefer to suspect the product is counterfeit. Let η denote the portion of the ordinary consumers, and hence $1 - \eta$ is the portion of the vigilant consumers. The average utility of a consumer for choosing the licit market is

$$u_{\theta o} = \eta \phi_g + (1 - \eta)\phi_f - p_o. \tag{3}$$

Let $a_\theta \in \mathcal{A}_c$ denote the consumer's action with type θ. The individual consumer's utility with type θ is

$$u_\theta = u_{\theta o} \mathbf{1}_{\{a_\theta = L\}} + u_{\theta b} \mathbf{1}_{\{a_\theta = I\}} + u_{\theta n} \mathbf{1}_{\{a_\theta = N\}}, \tag{4}$$

where $\mathbf{1}_{\{\bullet\}}$ is the indicator function. Since the population of all consumers is normalized to 1, and θ is uniformly distributed over $[0, 1]$, the net utility u_c of total consumers is the accumulative result of all u_θ:

$$u_c = \int_0^1 u_\theta d\theta, \tag{5}$$

which is independent of consumers' type θ.

2.2 Pricing Mechanisms in Licit and Illicit Market

The sale price is related to the amount of products in the market. We assume that the licit market's sale price has a linear relationship with the total amount of products in the licit market:

$$p_o = b_o - a_o(q_1 + q_2), \tag{6}$$

where b_o is the maximum price that may come from price controls, and a_o is the sale price elasticity. Illicit sellers may set the price for counterfeits based on their intentions. We assume that the illicit market's sale price comprises two terms:

$$p_b = \alpha p_o + (1 - \alpha)(b_b - a_b q_3). \tag{7}$$

The first term explains the price-taking strategy. The second term illustrates the price-setting strategy, in which illicit sellers have more control to set the price. We assume such control is linear in the production size q_3. b_b is the maximum price, and a_b is the price elasticity. $\alpha \in [0, 1]$ emphasizes the relative weight of two pricing strategies in the illicit market.

Since the market scale is 1, we can rescale the production size of each seller to the unit interval $[0, 1]$, as producing products more than one unit is not necessary. Therefore, the sale prices p_o and p_b vary in a range.

2.3 Sellers' Utility

Let D_o and D_b denote the licit and illicit market share, respectively. Note that D_o may contain deceptive counterfeits due to the counterfeit penetration. The real licit market share for the licit seller S_1 is $\frac{q_1}{q_1+q_2}D_o$. Likewise, the real licit market share for the illicit seller S_2 is $\frac{q_2}{q_1+q_2}D_o$. Thus, S_1's utility is given by

$$u_1 = \frac{q_1}{q_1+q_2}D_o p_o - c_1 q_1. \tag{8}$$

S_2's utility is similar, but he will face confiscation risks in the licit market. Let γ denote the seizure probability from authorities and s denote the confiscation penalty, then the utility of the seller S_2 is

$$u_2 = (1-\gamma)\frac{q_2}{q_1+q_2}D_o p_o - c_2 q_2 - \gamma s. \tag{9}$$

Due to the virtuality of the online market, illicit trades are hard to detect. Even when the authority discovers several illicit trades, illicit sellers can quickly close their virtual trade portals and start a new one elsewhere. Therefore, we set the confiscation probability for illicit sellers in the illicit market as 0. The utility of the illicit seller S_3 is

$$u_3 = D_b p_b - c_3 q_3. \tag{10}$$

3 Problem Formulation and Game Structure

In this section, we propose a games-in-games framework to formulate the counterfeiting problem. The action space of the seller S_i is $\mathcal{A}_i = \{q_i \mid 0 \le q_i \le 1\}$ for $i = 1, 2, 3$. A Nash game can capture the competition among three sellers. Each seller seeks to maximize his utility by choosing the optimal production size. Also, all sellers play a Stackelberg game with consumers. All consumers form the follower to make the purchase decisions and maximize their utility. The games-in-games framework is presented in Fig. 2.

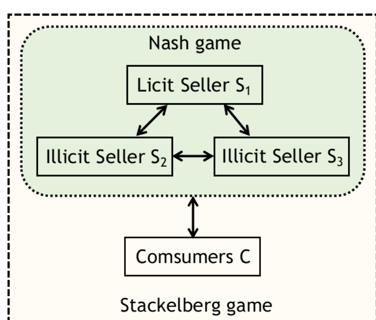


Fig. 2: The structure of the games-in-games framework.

3.1 Stackelberg Game

In the Stackelberg game, three sellers are the leader, and consumers are the follower. After observing the sale price p_o and p_b, every consumer decides her action to maximize her utility so that the consumers' net utility is maximized:

$$\mathcal{P}_c: \quad \max_{a_\theta \in \mathcal{A}_c} \int_0^1 u_\theta d\theta. \tag{11}$$

The leader's problem in the Stackelberg game is defined by another Nash game because three sellers compete with each other. The Stackelberg game can be represented by $\mathcal{G}_s := \{C, S_i, \mathcal{A}_c, \mathcal{A}_i, u_c, u_i\}$ for $i = 1, 2, 3$.

3.2 Nash Game

The interaction among thee sellers can be characterized as a Nash game, where each player aims to maximize his profit. We denote this strategic game by $\mathcal{G}_n := \{S_i, \mathcal{A}_i, u_i\}$, $i = 1, 2, 3$. The utility u_i, $i = 1, 2, 3$, are defined in (8)-(10). We denote the aggregated action space as $\mathcal{A} = \{(q_1, q_2, q_3) \mid 0 \leq q_1, q_2, q_3 \leq 1\}$. Therefore, $\mathcal{G}_n$ is a continuous game defined on the unit cube.

4 Analysis of Counterfeiting in Cyber Supply Chain

In this section, we present the equilibrium analysis of the counterfeiting problem. We first analyze the market share and derive its piecewise linear expression, which leads to a game with piecewise continuous utilities. Then we study algorithms to find the equilibrium solution by solving each seller's best response.

4.1 Market Share Analysis

The market share is a function of the solution to the consumers' problem (11). When solving for (11), we fix the production size q_1, q_2, q_3, and the sale price p_o, p_b. As u_c in (5) contains indicator functions, the consumers' problem (11) is equivalent to

$$\tilde{\mathcal{P}}_c: \quad \int_0^1 \max\{u_{\theta o}(\theta), u_{\theta b}(\theta), u_{\theta n}(\theta)\} d\theta, \tag{12}$$

The solution to (12) generates a series of critical points θ^*'s, which partitions the interval $[0, 1]$ into several subintervals. In each subinterval, $u_{\theta o}, u_{\theta b}$, or $u_{\theta n}$ is greater than the other two. Since the total market scale is 1, each subinterval length can be interpreted as the corresponding market share. Suppose that critical points yield an interval $[a, b]$ where $u_{\theta o}$ is greater than $u_{\theta b}$ and u_{θ_n}. This means that the licit market share $D_o = |b - a|$. Therefore, the market shares D_o and D_b refer to the subintervals where $u_{\theta o}$ and $u_{\theta b}$ is the largest element among $\{u_{\theta o}, u_{\theta b}, u_{\theta n}\}$, respectively.

Since $u_{\theta o}, u_{\theta b}$, and $u_{\theta n}$ are linear in θ, we can find the critical points θ_{on}, θ_{bn}, and θ_{ob} by solving three equations: $u_{\theta o} = u_{\theta n}$ gives $\theta_{on} = \frac{k - p_o}{ka(1-\eta)}$; $u_{\theta b} = u_{\theta n}$

gives $\theta_{bn} = \frac{k-p_b}{ka}$; $u_{\theta o} = u_{\theta b}$ gives $\theta_{ob} = \frac{p_o - p_b}{\eta ka}$. These critical points form intervals that represent the market shares. Note that the market share can vary quite often due to the change of the production size q_i. We make two assumptions to simplify our analysis.

First, we assume that the illicit market sale price is always less than the licit market sale price, *i.e.*, $p_b \leq p_o$. There is no reason for a consumer to purchase a counterfeit if she knows it is more expensive than a genuine product.

Second, we assume that the illicit market is always attractive to some portion of the consumers, but it will never attract the entire consumer body to purchase counterfeits. The former part of the assumption indicates that the illicit market is preferable for the consumers with small θ (near 0). It can happen because the first assumption suggests that the counterfeit price is always less than the real goods. Mathematically, it means that $k - p_{b,\max} \geq 0$. The latter part of the assumption is natural as we never expect all consumers to go to the illicit market to buy counterfeits. Mathematically, it is equivalent to the critical point $\theta_{bn} < 1$, which is illustrated in the following proposition.

Proposition 1. *If $k - ka \leq p_{b,\min}$, then θ_{bn} is guaranteed to be within* $[0, 1]$.

With the assumptions above, there are three cases to discuss to find the market shares. Each case corresponds to a sub-region in the aggregated action space $\mathcal{A}$.

- Region I ($\mathcal{R}_1$), *the illicit market monopoly region*, where only the illicit market share is positive. We have $\theta_{on} \leq \theta_{bn}$; $D_o = 0$ and $D_b = \theta_{bn}$.
- Region II ($\mathcal{R}_2$), *the partial competition region*, where consumers have three actions to take. We have $\theta_{on} > \theta_{bn}$ and $\theta_{on} < 1$; $D_o = \theta_{on} - \theta_{ob}$ and $D_b = \theta_{ob}$.
- Region III ($\mathcal{R}_3$), *the pure competition region*, where consumers only have two available actions (I and L), because buying nothing yields the least utility. We have $\theta_{on} > \theta_{bn}$ and $\theta_{on} \geq 1$; $D_o = 1 - \theta_{ob}$ and $D_b = \theta_{ob}$.

Therefore, the aggregated action space $\mathcal{A}$ is partitioned into three sub-regions by $\theta_{on} = \theta_{bn}$ and $\theta_{on} = 1$. Let $q := \begin{bmatrix} q_1 \ q_2 \ q_3 \end{bmatrix}^T \in \mathbb{R}^3$. We write

$$\theta_{on} = \theta_{bn} \;\Rightarrow\; A_1 q + b_1 = 0, \quad \theta_{on} = 1 \;\Rightarrow\; A_2 q + b_2 = 0.$$

where $A_1 = \begin{bmatrix} -a_o(1-\alpha(1-\eta)) \ -a_o(1-\alpha(1-\eta)) \ (1-\alpha)(1-\eta)a_b \end{bmatrix}$ and $b_1 = -(\alpha(1-\eta)-1)b_o - (1-\alpha)(1-\eta)b_b - \eta k$; $A_2 = \begin{bmatrix} -a_o \ -a_o \ 0 \end{bmatrix}$ and $b_2 = b_o - k + (1-\eta)ka$. Therefore, three sub-regions can be characterized as

$$\begin{aligned} \mathcal{R}_1 &= \{q \mid A_1 q + b_1 \geq 0, 0 \leq q \leq 1\}, \\ \mathcal{R}_2 &= \{q \mid A_1 q + b_1 \leq 0, A_2 q + b_2 \geq 0, 0 \leq q \leq 1\}, \\ \mathcal{R}_3 &= \{q \mid A_2 q + b_2 \leq 0, 0 \leq q \leq 1\}. \end{aligned} \tag{13}$$

Fig. 3 shows an example of the partition of $\mathcal{A}$.

Remark 1. We denote $\mathcal{C}_{ij}$ as the plane that separates $\mathcal{R}_i$ and $\mathcal{R}_j$. $\mathcal{C}_{12} = \{q \mid A_1 q + b_1 = 0, 0 \leq q \leq 1\}$ is called the *profit plane* as $\mathcal{C}_{12}$ determines whether D_o is zero or not. $\mathcal{C}_{23} = \{q \mid A_2 q + b_2 = 0, 0 \leq q \leq 1\}$ is called the *growth rate switch plane* as the growth rate of D_o changes when crossing $\mathcal{C}_{23}$.

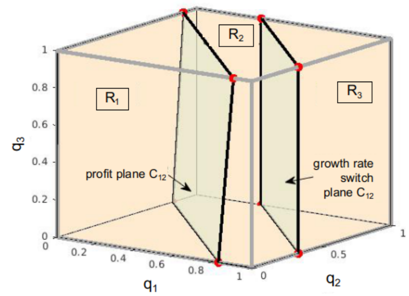


Fig. 3: Partition sketch of the aggregated action space $\mathcal{A}$.

Based on (13), we obtain the expression of the market shares D_o and D_b:

$$D_o = \begin{cases} 0 & q \in \mathcal{R}_1 \\ \frac{k-p_o}{ka(1-\eta)} - \frac{p_o-p_b}{\eta ka} & q \in \mathcal{R}_2 \\ 1 - \frac{p_o-p_b}{\eta ka} & q \in \mathcal{R}_3 \end{cases}, \quad D_b = \begin{cases} \frac{k-p_b}{ka} & q \in \mathcal{R}_1 \\ \frac{p_o-p_b}{\eta ka} & q \in \mathcal{R}_2 \cup \mathcal{R}_3 \end{cases}, \tag{14}$$

where p_o and p_b are defined in (6) and (7), respectively. Therefore, D_o and D_b are piecewise continuous, which results in piecewise continuous utility u_i, $i = 1, 2, 3$. The game $\mathcal{G}_n$ is a three-player Nash game with piecewise continuous utility functions.

4.2 Best Response Functions

The definition of seller S_i's best response is as follows.

Definition 1 (Best response). *For the seller S_i with action q_i, his best response is defined by $BR_i(q_{-i}) := \{v \in \mathcal{A}_i \mid u_i(v, q_{-i}) \geq u_i(w, q_{-i}) \ \forall w \in \mathcal{A}_i\}$, where $i \in \{1, 2, 3\}$ and $-i := \{1, 2, 3\} \backslash \{i\}$.*

Best Response of Seller S_3 Since D_b varies differently in $\mathcal{R}_1$ and $\mathcal{R}_2 \cup \mathcal{R}_3$ from (14), the utility u_3 has two distinct expressions in the corresponding regions. We first find the unconstrained maximizers of u_3 in $\mathcal{R}_1$ and $\mathcal{R}_2 \cup \mathcal{R}_3$, respectively, and then compute the best response of S_3 by comparing the utility values at the boundary points and at the unconstrained maximizers.

In $\mathcal{R}_1$, we have $u_3 = \frac{k-p_b}{ka} p_b - c_3 q_3$, which is quadratic and concave in q_3. The unconstrained maximizer in $\mathcal{R}_1$ is:

$$q^*_{3,R1}(q_1, q_2) = \arg\max_{q_3} u_3 = \frac{b_b}{a_b} + \frac{\alpha p_o}{(1-\alpha)a_b} - \frac{k}{2(1-\alpha)a_b} - \frac{c_3 ka}{2(1-\alpha)^2 a_b^2}. \tag{15}$$

Likewise, in $\mathcal{R}_2 \cup \mathcal{R}_3$, we have $u_3 = \frac{p_o-p_b}{\eta ka} p_b - c_3 q_3$. The unconstrained maximizer in this region is

$$q^*_{3,R23}(q_1, q_2) = \frac{b_b}{a_b} + \frac{\alpha p_o}{(1-\alpha)a_b} - \frac{p_o}{2(1-\alpha)a_b} - \frac{c_3 \eta ka}{2(1-\alpha)^2 a_b^2}. \tag{16}$$

Note that the given pair (q_1, q_2) affects which sub-region the seller S_3 will stay. For example, if $(q_1, q_2) = (0, 0)$, S_3 must stay in $\mathcal{R}_1$. If $(q_1, q_2) = (1, 1)$, S_3 can only stay in $\mathcal{R}_2 \cup \mathcal{R}_3$. Also, the separating plane $\mathcal{C}_{12}$ and the box constraints are also critical to determine which sub-region the seller S_3 is in. We propose Algorithm 1 to find the best response of S_3. For simplicity we write $u_3(q_3) := u_3(q_1, q_2, q_3)$ as (q_1, q_2) are parameters when computing best response.

Algorithm 1: Calculate the best response $BR_3(q_1, q_2)$ of the seller S_3.

input: q_1, q_2
$q_t \leftarrow (-b_1 - A_{1,1}q_1 - A_{1,2}q_2)/A_{1,3}$; `// compute crossing point on` $\mathcal{C}_{12}$
if $q_t \leq 0$ **then**
  $BR_3(q_1, q_2) \leftarrow \arg\max\{u_3(0), u_3(1), u_3(q^*_{3,R1})\}$; `//` S_3 `is in` $\mathcal{R}_1$
else if $q_t \geq 1$ **then**
  $BR_3(q_1, q_2) \leftarrow \arg\max\{u_3(0), u_3(1), u_3(q^*_{3,R23})\}$; `//` S_3 `is in` $\mathcal{R}_2 \cup \mathcal{R}_3$
else
  $\tilde{q}^*_{3,R1} \leftarrow \arg\max\{u_3(0), u_3(q_t), u_3(q^*_{3,R1})\}$; `// maximizer in` $\mathcal{R}_1$
  $\tilde{q}^*_{3,R23} \leftarrow \arg\max\{u_3(q_t), u_3(1), u_3(q^*_{3,R23})\}$; `// maximizer in` $\mathcal{R}_2 \cup \mathcal{R}_3$
  $BR_3(q_1, q_2) \leftarrow \arg\max\{u_3(\tilde{q}^*_{3,R1}), u_3(\tilde{q}^*_{3,R23})\}$
end

Note that $\arg\max\{\cdot\}$ is the argument of the maximum element. For example, if $f(a) > f(b)$, then $a = \arg\max\{f(a), f(b)\}$.

Best Response of Seller S_1 For the seller S_1, as D_o has distinct definitions in $\mathcal{R}_1, \mathcal{R}_2$ and $\mathcal{R}_3$, we have three different cases for the utility u_1. Since $D_o = 0$ in $\mathcal{R}_1$, the utility becomes $u_1 = -c_1 q_1$. Thus, the maximizer of u_1 in $\mathcal{R}_1$ is $q^*_{1,R1} = 0$. When entering $\mathcal{R}_2$, D_o becomes positive, and the utility is

$$u_1 = \frac{q_1}{q_1 + q_2}\left(\frac{k - p_o}{ka(1-\eta)} - \frac{p_o - p_b}{\eta k a}\right) p_o - c_1 q_1.$$

To find the maximizer of u_1 in $\mathcal{R}_2$, we note that both the sale price p_o and the market share D_o are positive in $\mathcal{R}_2$. Although the concavity of u_1 is not guaranteed, the positivity of the two quantities allows us to use the gradient accent method to find the maximizer. We arrive at the following theorem.

Theorem 1. *Suppose $f(x)$ is a concave and quadratic function and let $g(x) = \frac{x}{x+a}$ where $a > 0$. Suppose that $f(x)$ is positive on $[b, c]$ where $b > 0$, then the maximizer of $f(x)g(x)$ on $[b, c]$ is either at the boundary or a stationary point that satisfies the first-order condition, and the maximizer is unique.*

Proof. See Appendix A.

From (6) and (14), we know that in $\mathcal{R}_2$ and $\mathcal{R}_3$, D_o and p_o are both positive and linear in q_1. D_o has a positive coefficient for q_1 while p_o has a negative one.

Algorithm 2: Calculate the best response $BR_1(q_2, q_3)$ of the seller S_1.

```
input: q_2, q_3
Compute crossing points for each sub-region ;
 // I_1 = [0, a], I_2 = [a, b], I_3 = [b, 1]
for i ← 1 to 3 do
    u_1 ← λ D_{o,i} p_o − c_1 q_1 ; // D_{o,i} refers to D_o in R_i
    if i = 1 then
        q*_{1,Ri} ← 0
    else
        Gradient ascent: q*_{1,Ri} ← arg max_{q_1 ∈ I_i} u_1(q_1) ;
    end
end
BR_1(q_2, q_3) ← arg max{u_1(q*_{1,R1}), u_1(q*_{1,R2}), u_1(q*_{1,R3})} ;
```

So $D_o p_o$ is positive, quadratic, and concave in q_1. Using Theorem 1, we can get the unique maximizer of u_1 in $\mathcal{R}_2$ and $\mathcal{R}_3$ using the gradient ascent method.

Fig. 3 helps visualize how to compute the crossing points of each sub-region. When fixing (q_2, q_3) and varying q_1 from 0 to 1, S_1 will pass $\mathcal{R}_1$, $\mathcal{R}_2$ and $\mathcal{R}_3$ in turn, leaving two crossing points denoted as a and b. The crossing points generate three intervals $\mathcal{I}_1 = [0, a]$, $\mathcal{I}_2 = [a, b]$, and $\mathcal{I}_3 = [b, 1]$. $q_1 \in \mathcal{I}_i$ indicates that S_1 is in the sub-region $\mathcal{R}_i$, $i = 1, 2, 3$. Note that when q_2 is close to 0 (or 1), we have $b = 1$ (or $a = 0$), which means S_1 will not appear in $\mathcal{R}_3$ (or $\mathcal{R}_1$).

Best Response of Seller S_2 The seller S_2's utility u_2 has almost the same structure as u_1. Therefore, Algorithm 2 can be applied to find the best response of S_2 by substituting u_1 with u_2. The computation of crossing points of each sub-region for S_2 is also similar. We fix (q_1, q_3) and vary q_2 from 0 to 1, and then compute the crossing point a, b and the related intervals $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$.

4.3 Iterative Algorithm

The definition of the Nash equilibrium of three sellers is as follows.

Definition 2 (Nash equilibrium). *The Nash equilibrium solution to the game $\mathcal{G}_n$ is a set of strategies $(q_1^*, q_2^*, q_3^*) \in \mathcal{A}$ such that for all $q_i \in \mathcal{A}_i$, $i \in \{1, 2, 3\}$,*

$$u_i(q_i^*, q_{-i}^*) \geq u_i(q_i, q_{-i}^*).$$

The equilibrium of a Nash game can be found by the intersection of all players' best responses. We propose Algorithm 3 based on iterative methods to find the equilibrium of the game $\mathcal{G}_n$. The existence of the equilibrium can be demonstrated by simulations. A notable result in our problem is that the best response of the seller S_3 is a point-to-point mapping provided that some conditions are satisfied. We arrive at the following results.

Theorem 2. *The seller S_3 in the Nash game $\mathcal{G}_n$ has a point-to-point best response provided that the inequalities* (17)-(18) *hold:*

$$\frac{\eta k + (1-\eta)(1-\alpha)b_b}{1-\alpha(1-\eta)} \leq \frac{3-\eta}{2}k + \frac{(1-\eta)c_3 ka}{2(1-\alpha)a_b}, \tag{17}$$

$$\frac{\eta k + (1-\eta)(1-\alpha)(b_b - a_b)}{1-\alpha(1-\eta)} \geq \frac{2\eta k}{1+\eta} - \frac{\eta(1-\eta)c_3 ka}{(1+\eta)(1-\alpha)a_b}. \tag{18}$$

Besides, the best response is continuous.

Proof. See Appendix B.

Numerical results corroborate that the best responses of the players S_1 and S_2 are point-to-point mappings under suitable parameters. The fixed point iterative method can be used to find the pure strategy equilibrium of the problem given an initial point $q_0 := (q_{1,0}, q_{2,0}, q_{3,0})$.

Algorithm 3: Iterative method to find Nash equilibrium.

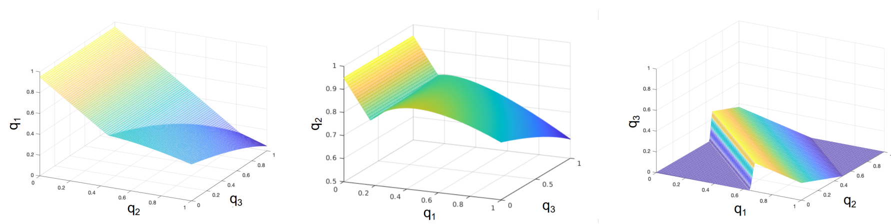
```
input: q_0 = (q_{1,0}, q_{2,0}, q_{3,0}), ϵ, k_max
k ← 0 ; // iteration counter
while k < k_max do
    /* we denote br_k(q) as best responses of player S_i          */
    BR_k(q_k) ← [br_1(q_k)  br_2(q_k)  br_3(q_k)]^T ; // form aggregated best
     response
    q_{k+1} ← BR_k(q_k) ;
    if ||q_{k+1} − q_k|| ≤ ϵ then
        break ;
    end
    q_k ← q_{k+1} ;
    k ← k + 1;
end
q_nash ← q_{k+1}.
```

5 Case Studies and Simulations

In this section, we use case studies to quantify the equilibrium strategy using the designed algorithms. Consider a licit and an illicit market characterized by $b_o = 10, a_o = 2$ and $b_b = 7.5, a_b = 1.5$. The parameter $\alpha = 0.6$ indicates the weight of the illicit seller's price-taking strategy. The consumers are parameterized by $a = \frac{1}{3}$ and $k = 9.5$. The production costs for three sellers are set to $c_1 = 4$ and $c_2 = c_3 = 2$. The cheap production cost differentiates the counterfeits with the genuine product and indicates the counterfeit's low quality. The confiscation probability and the penalty are $\gamma = 0.2$ and $s = 3$. The portion of non-vigilant consumers in the licit market is $\eta = 0.7$.

5.1 Best Response and Equilibrium Strategy

The best responses of S_1, S_2, and S_3 are presented in Fig. 4. The best response of S_3 is continuous, as proved before. Simulations also show that the other two players' best responses are point-to-point mappings. The equilibrium in this case is $(q_1, q_2, q_3) = (0.44, 0.82, 0.09)$. We notice that S_1's best response production increases with q_3 when there are few deceptive counterfeits. This shows that in the competition with S_3, S_1 can always keep a low sale price to attract more consumers from the illicit market. The best response of S_3 corroborates the claim. It decreases when q_1 increases (for large q_1), which indicates that non-deceptive counterfeits are no longer attractive to consumers as the licit market price goes down. This is common in the cyber supply chain because sellers' anonymity provides less useful information to products and makes consumers more price sensitive. However, deceptive counterfeits can significantly erode the licit market share and affect S_1's best production strategies. We see that S_1's best response decreases with q_2, and whatever the condition is, the best-response production of S_2 is large. This is because S_2 takes advantage of consumers' trust in the licit market and the low counterfeit production cost to steal the licit market share. The anonymity and the reduced structure in the cyber supply chain make the situation even worse, accounting for the considerable value of q_2 in the equilibrium. The fact indicates the deceptive counterfeits can pose more severe threats to the cyber supply chain security than non-deceptive counterfeits.



(a) Best response of S_1. (b) Best response of S_2. (c) Best response of S_3.

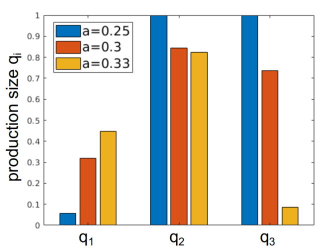
Fig. 4: All sellers' best responses are continuous and point-to-point mappings.

Remark 2. We notice that the simulation shows the total production size $(q_1 + q_2 + q_3)$ not equal to the total demand (1) at the equilibrium. This result can be viewed as the outcome of the competitive production planning process. The competition in production planning can lead to oversupply and thus the inefficiency of the equilibrium. The phenomenon of oversupply aligns with several observations made by works in management operations. Christensen in [6] has discussed the oversupply issues of rigid disk drivers due to competitions. The oversupply can also be interpreted by the asymmetric position of sellers and buyers. In economics, the perfect market equilibrium is characterized by the best response of sellers and buyers; *i.e.*, they are in the symmetric position. The impact on each other eventually leads to the market equilibrium with supply equal to demand. In our framework, the consumers are the follower of a Stackelberg game instead of a Nash game player. The sellers can anticipate the consumers' demand, but the consumers follow the price determined by the sellers' production decision-making. Three sellers plan their optimal production volumes by mainly considering the competition among each other.

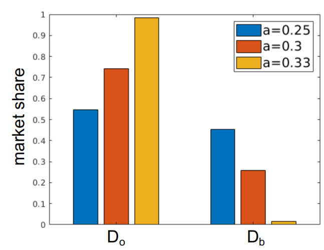
5.2 Discussion on Parameter Sensitivity

Some parameters are critical for the player's performance and the equilibrium. We discuss and visualize their impact on the equilibrium using simulations.

Valuation Elasticity a Consumers' tolerance on counterfeits is reflected by a. The larger a is, the fewer consumers are willing to tolerate counterfeits, leading to a decreasing profit and production size to the illicit sellers. We let $a = 0.25, 0.3, 0.33$, respectively, leaving other parameters unchanged. The equilibria when $a = 0.3$ and $a = 0.33$ are $(q_1, q_2, q_3) = (0.31, 0.85, 0.73)$ and $(0.44, 0.82, 0.09)$, respectively. As a progresses, counterfeits become less attractive. The production volumes of S_1 and S_2 are suppressed as shown in Fig. 5.



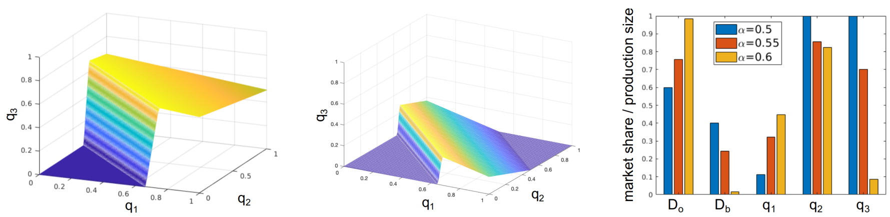
(a) Production size q_i in different valuation elasticity a.



(b) Market shares D_o and D_b in different valuation elasticity a.

Fig. 5: Increasing a can diminish the production size of illicit sellers and the illicit market share.

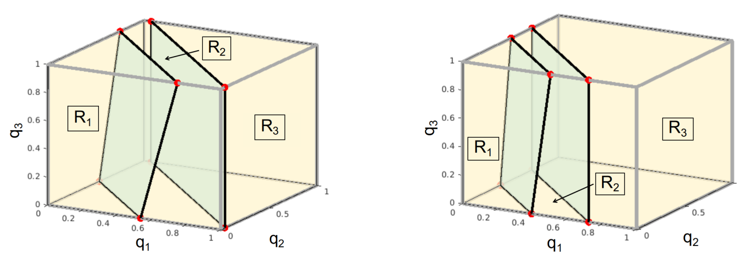
Weight of Price-Taking Strategy α Large α means that the illicit seller is more willing to follow the licit market sale price than set the price. As Fig. 6 indicates, large α corresponds to a small best-response production of S_3. S_3 gradually loses his price advantage as α increases. The consequence of large α is that the licit and illicit markets have comparable sale prices. Consumers naturally prefer the licit market when the prices are similar. Although there may be deceptive counterfeits in the licit market, the average utility of purchasing in the licit market can be higher than that in the illicit market, depending on the consumers' belief η. This explains why S_3 loses the market share under large α.



(a) Best response of S_3 when $\alpha = 0.55$. (b) Best response of S_3 when $\alpha = 0.6$. (c) Market shares and equilibrium under different α.

Fig. 6: Large α can effectively diminish the illicit seller S_3's production, and free more market share to the licit market.

Consumers' Belief in Licit Market $\boldsymbol{\eta}$ We see that $\eta < 1$ tilts the profit plane $\mathcal{C}_{12}$, which amplifies the illicit market's impact on the licit market by reducing the consumers' average utility in the licit market. The licit market has to attract more consumers to maintain the same consumers' utility than $\eta = 1$. Also, increasing η helps reduce the size of $\mathcal{R}_1$ as Fig. 7 shows. Smaller $\mathcal{R}_1$ allows S_1 to establish a positive market share D_o more easily. As the total market share is fixed, larger η decreases illicit sellers' profit and crushes counterfeits production. The simulation corroborates the conclusion. The market equilibrium is $(q_1, q_2, q_3) = (0.05, 1.0, 1.0)$ when $\eta = 0.5$, which yields $D_o = 0.51$ and $D_b = 0.49$. When η increases to 0.7, the equilibrium becomes $(0.44, 0.82, 0.09)$, which gives a market share $D_o = 0.98$ and $D_b = 0.02$. Two illicit sellers' production capacities are reduced, and the illicit market share is also suppressed. The utility of the licit seller S_1 increases from $u_1 = 0$ to $u_1 = 0.8$.

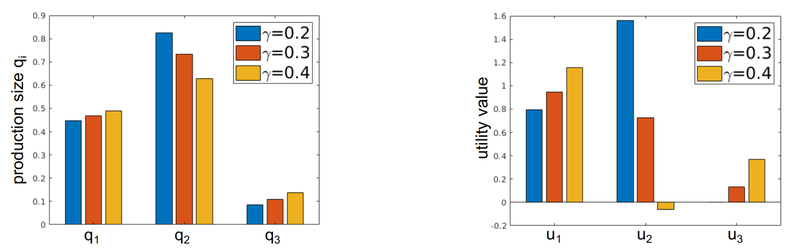


(a) $\eta = 0.5$. (b) $\eta = 0.7$.

Fig. 7: The partition of the aggregated action space $\mathcal{A}$. Larger η yields smaller $\mathcal{R}_1$, which helps S_1 take positive market share more easily.

Confiscation Probability $\boldsymbol{\gamma}$ We discuss the impact of γ because the penalty s is a constant term in S_2's utility. As Fig. 8 shows, a large γ reduces the illicit seller

S_2's production size and profit. S_1's production size and profit are increasing with γ as expected. However, S_1 does not entirely absorb the market share freed by S_2. S_3 also takes some of it, which is because of the *price advantage*: the illicit market sale price is less than the licit price, so counterfeits are always attractive to some portion of consumers. Notably, deceptive counterfeit penetration is more severe than expected. When $\gamma = 0.2$, the amount of deceptive counterfeits reaches $q_2 = 0.82$. Counterfeiting is rampant when the inspection is loose, and the deceptive counterfeit penetration scale is much larger than the non-deceptive counterfeit trafficking in the illicit market. This is particularly true in the cyber supply chain because the wide availability and online sellers' anonymity make inspections much difficult. Note that S_2's utility is negative when $\gamma = 0.4$. It means the confiscation risk is greater than the revenue of selling deceptive counterfeits, providing an effective anti-counterfeit strategy.



(a) Production size q_i under different γ. (b) Utility u_i under different γ.

Fig. 8: Large γ can restrain the illicit seller S_2's production and free more market share. However, the freed market share is taken by both sellers S_1 and S_3.

5.3 Anti-Counterfeit Strategies

The special features of the cyber supply chain are characterized by parameters. The parameter sensitivity analysis in Section 5.2 provides insights on how to suppress the profit and illicit sellers' production. Based on this, we propose strategies to mitigate counterfeiting in the cyber supply chain ecosystem.

First, the strategies that improve the consumers' faith in the licit market (increasing η) and reduce their tolerance on counterfeits (increasing a) can weaken counterfeiting problems in the cyber supply chain. Based on the cyber platform, actions such as third-party cyber insurance for products and digital authentications from the market organizer will help.

Second, the strategies that increase α can help diminish counterfeit production size. Large α indicates less power of illicit sellers in setting the illicit market sale price. One way to enlarge α is to attract more licit sellers to participate in the cyber supply chain. In this way, the illicit sellers' autonomy on price-making will be weakened, and they are forced to follow the licit market sale price. Optimizing the cyber supply chain process for licit sellers also contributes to improving α. For example, more efficient delivery and economical production processes can make genuine products more competitive, so that licit sellers seize more initiative in the price-making stage.

Third, improving confiscation probability γ can suppress the deceptive counterfeits. Deceptive counterfeits can jeopardize the licit market and licit supply chain by taking more market share and disrupting the market order. Although it is potentially hard to trace the deceptive counterfeits in the cyber supply chain, advanced techniques can be considered, such as using RFID chip nad QR code for verification, adopting blockchain technologies to track product life cycle.

6 Conclusion

In this work, we have investigated the equilibrium and anti-counterfeit strategies on counterfeiting activities in the cyber supply chain ecosystem by establishing a games-in-games framework. We first analyze the features of the cyber supply chain and use the game-theoretic approach to capture the interactions in counterfeiting activities. A Nash game and a Stackelberg game are then formulated to yield a games-in-games framework. The developed algorithms are used to calculate the best responses of the sellers and the market equilibrium. Case studies investigate different sellers' behaviors under different scenarios. Based on the analysis and numerical experiments, three anti-counterfeit strategies are proposed to restrain counterfeiting activities. The framework also opens a way to understand the complex interactions of counterfeiting in the cyber supply chain. Future work will explore the equilibrium in incomplete information scenarios. Information asymmetry may affect the market equilibrium, which is crucial to combat counterfeiting in the cyber supply chain ecosystem. Limited information will also impact the available anti-counterfeit strategies for both authorities and licit suppliers.

A Proof of Theorem 1

Let $f(x) = px^2 + qx + r$. The concavity of f indicates that $p < 0$, and let $h(x) = f(x)g(x)$. Then

$$h''(x) = 2pg(x) + f(x)\frac{-2a}{(x+a)^3} + \frac{2a}{(x+a)^2}(2px+q).$$

Note that $g(x)$ is concave and increasing on $[b, c]$. We discuss three possibilities.

- $f(x)$ is increasing on $[b, c]$, *i.e.*, $-\frac{q}{2p} \geq c$. Since $f(x)$ and $g(x)$ are both increasing and positive on $[b, c]$, thus $x^* = \arg\max h(x) = c$.
- $f(x)$ is decreasing on $[b, c]$, *i.e.*, $-\frac{q}{2p} \leq b$. We check the Hessian of $h(x)$. The first two term are clearly negative. As $f(x)$ is decreasing, we have $f'(x) = 2px+q < 0$ on $[b, c]$. Therefore, the Hessian $h''(x) < 0$ and $h(x)$ is concave on $[b, c]$. The maximizer of $h(x)$ can characterized by the first-order condition $h'(x_{foc}) = 0$. Thus x^* is the argument of $\max\{h(b), h(c), h(x_{foc})\}$.

– $f(x)$ is both increasing and decreasing on $[b,c]$, *i.e.*, $b < -\frac{q}{2p} < c$. We split $[b,c]$ into two subintervals $[b,-\frac{q}{2p}]$ and $(-\frac{q}{2p},c]$. In the first interval, $h(x)$ is increasing. In the second interval, $h(x)$ is concave. Note that $h(x)$ is continuously differentiable on $[b,c]$, which means $h'(x)$ is continuous. Since

$$h'(-\frac{q}{2p}) = f(-\frac{q}{2p})g'(-\frac{q}{2p}) > 0,$$

$h'(x)$ is positive in a small neighborhood of $x = -\frac{q}{2p}$, which means $h(x)$ is still increasing in that small neighborhood. Therefore, we obtain that the maximizer is either $x = c$ or the point which satisfies the first-order condition, *i.e.*, x^* is the argument of $\max\{h(c), h(x_{foc})\}$.

Since $h'(x)$ is continuous and is nonzero constant on any subintervals of $[b,c]$, the uniqueness of the maximizer is guaranteed.

B Proof of Theorem 2

From (14), The nonsmoothness of D_b occurs when q_3 crosses $\mathcal{C}_{12}$. Let $\mathcal{I}_1$ and $\mathcal{I}_{23}$ denote the interval such that $D_b = \frac{k-p_b}{ka}$ when $q_3 \in \mathcal{I}_1$ and $D_b = \frac{p_o-p_b}{\eta ka}$ when $q_3 \in \mathcal{I}_{23}$. Note that $\mathcal{I}_1$ and $\mathcal{I}_{23}$ are parameterized by q_1 and q_2. We write $\mathcal{I}_{23} = [0, q_{3,s}]$ and $\mathcal{I}_1 = [q_{3,s}, 1]$. When $q_{3,s} = 0$ or 1, $\mathcal{I}_{23}$ or $\mathcal{I}_1$ is empty; when $q_{3,s} \in (0,1)$, both $\mathcal{I}_1$ and $\mathcal{I}_{23}$ are nonempty. We call $q_{3,s}$ the *crossing point.*

When one of $\mathcal{I}_1$ and $\mathcal{I}_{23}$ is empty, D_b is smooth on the entire $[0,1]$. As the utility function u_3 is concave, the maximizer is unique. When $\mathcal{I}_1$ and $\mathcal{I}_{23}$ are both nonempty, u_3 comprises two concave and quadratic functions $u_{3,1}, u_{3,23}$ on $\mathcal{I}_1$ and $\mathcal{I}_{23}$. Clearly, the concavity of u_3 is not guaranteed.

Let $\mathcal{W} = \{(q_1,q_2,q_3) \mid 0 \le q_1, q_2 \le 1, q_3 = 0\}$, and $\operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}$ be the projection of $\mathcal{C}_{12}$ onto $\mathcal{W}$. Note that if $\eta < 1$, the profit plane $\mathcal{C}_{12}$ is not parallel to the q_3 axis, and hence the projection forms a closed polytope: $\operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12} = \{(q_1,q_2) \mid (q_1,q_2,q_3) \in \mathcal{C}_{12}, \forall q_3 \in [0,1]\}$. When $(q_1,q_2) \in \operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}$, $\mathcal{I}_1$ and $\mathcal{I}_{23}$ are both nonempty. When $(q_1,q_2) \notin \operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}$, either $\mathcal{I}_1$ or $\mathcal{I}_{23}$ is empty. Next, we let $q_{3,u}^*$ and $q_{3,u}^{**}$ be the unconstrained maximizers of $u_{3,1}$ and $u_{3,23}$, respectively. Let q_3^* be the maximizer of u_3 in $[0,1]$. To guarantee the uniqueness of q_3^*, we set q_3^* as the crossing point $q_{3,s}$. The following inequalities must hold:

$$q_{3,u}^* \le q_{3,s}, \quad q_{3,u}^{**} \ge q_{3,s} \quad \forall (q_1,q_2) \in \operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}$$

Further simplification tells for all $(q_1,q_2) \in \operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}$, we have

$$p_o \le \frac{3-\eta}{2}k + \frac{(1-\eta)c_3ka}{2(1-\alpha)a_b}, \quad p_o \ge \frac{2\eta k}{1+\eta} - \frac{\eta(1-\eta)c_3ka}{(1+\eta)(1-\alpha)a_b}$$

Since $\operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}$ is closed, $p_{o,\min}$ and $p_{o,\max}$ exist. By taking these two values into the inequalities above, we obtain the inequalities (17)-(18).

To prove the continuity, it is clear that $\operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12} \subset \mathcal{W}$. For the region $\{(q_1,q_2,q_3) \mid (q_1,q_2) \in \operatorname{proj}_{\mathcal{W}} \mathcal{C}_{12}, q_3 \in [0,1]\}$, the best response is the crossing point $q_{3,s}$. All the crossing points form the plane $\mathcal{C}_{12}$, which is continuous in (q_1, q_2). For the region $\{(q_1, q_2, q_3) \mid (q_1, q_2) \in \mathcal{W} \setminus \mathrm{proj}_{\mathcal{W}} \mathcal{C}_{12}, q_3 \in [0, 1]\}$, the best response is either 0 or 1 or the unconstrained minimizer calculated by (15) or (16). All of them are continuous in (q_1, q_2). This proves the continuity of the best response of the seller S_3.

References

1. Association, I.T., et al.: Addressing the sale of counterfeits on the internet (2013)
2. Beamon, B.M.: Supply chain design and analysis:: Models and methods. International journal of production economics **55**(3), 281–294 (1998)
3. Berman, B.: Strategies to detect and reduce counterfeiting activity. Business Horizons **51**(3), 191–199 (2008)
4. Buratto, A., Grosset, L., Zaccour, G.: Strategic pricing and advertising in the presence of a counterfeiter. IMA Journal of Management Mathematics **27**(3), 397–418 (2016)
5. Cho, S.H., Fang, X., Tayur, S.: Combating strategic counterfeiters in licit and illicit supply chains. Manufacturing & Service Operations Management **17**(3), 273–289 (2015)
6. Christensen, C.: Patterns in the evolution of product competition. European Management Journal **15**(2), 117–127 (1997)
7. deKieffer, D.E.: The internet and the globalization of counterfeit drugs. Journal of Pharmacy Practice **19**(3), 171–177 (2006)
8. Eser, Z., Kurtulmusoglu, B., Bicaksiz, A., Sumer, S.I.: Counterfeit supply chains. Procedia economics and finance **23**, 412–421 (2015)
9. Grossman, G.M., Shapiro, C.: Foreign counterfeiting of status goods. The Quarterly Journal of Economics **103**(1), 79–100 (1988)
10. Guin, U., Forte, D., Tehranipoor, M.: Anti-counterfeit techniques: From design to resign. In: 2013 14th International workshop on microprocessor test and verification. pp. 89–94. IEEE (2013)
11. Guin, U., Huang, K., DiMase, D., Carulli, J.M., Tehranipoor, M., Makris, Y.: Counterfeit integrated circuits: A rising threat in the global semiconductor supply chain. Proceedings of the IEEE **102**(8), 1207–1228 (2014)
12. Li, F., Yi, Z.: Counterfeiting and piracy in supply chain management: theoretical studies. Journal of Business & Industrial Marketing (2017)
13. Min, H., Zhou, G.: Supply chain modeling: past, present and future. Computers & industrial engineering **43**(1-2), 231–249 (2002)
14. Radón, A.: Counterfeit luxury goods online: an investigation of consumer perceptions. International Journal of Marketing Studies **4**(2), 74 (2012)
15. Shen, Z.: Integrated supply chain design models: a survey and future research directions. Journal of industrial and management optimization **3**(1), 1 (2007)
16. Taleizadeh, A.A., Noori-daryan, M., Tavakkoli-Moghaddam, R.: Pricing and ordering decisions in a supply chain with imperfect quality items and inspection under buyback of defective items. International Journal of Production Research **53**(15), 4553–4582 (2015)
17. The Council of Economic Advisers: The role of opioid prices in the evolving opioid crisis. https://www.whitehouse.gov/cea/research/ (2019), online; accessed 10 August 2020
18. Zhang, J., Hong, L.J., Zhang, R.Q.: Fighting strategies in a market with counterfeits. Annals of Operations Research **192**(1), 49–66 (2012)